EXACT COSMOLOGICAL SOLUTIONS OF NONLINEAR $F(R)$-GRAVITY

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We report on the cited papers refs. 1 - 18 from the following points of view: What do we exactly know about solutions when no exact solution (in the sense of "solution in closed form") is available? In which sense do these solutions possess a singularity? In which cases do conformal relations and/or dimensional reductions simplify the deduction? Furthermore, we outline some open questions worth of being studied in future research.

1 Singularity Theorem

In ref. 1 the following simple type of singularity theorems was discussed: The coordinates $t,\ x,\ y,\ z$ shall cover all the reals, and $a(t)$ shall be an arbitrary strictly positive monotonously increasing smooth function defined for all real values t, where "smooth" denotes "C^∞-differentiable". Then it holds
Lemma 1: The Riemannian space defined by

$$ds^2 = dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

is geodesically complete.

This fact is well-known and easy to prove; however, on the other hand it holds for the same class of functions $a(t)$
Lemma 2: The Pseudoriemannian space–time defined by

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \qquad (1)$$

is light–like geodesically complete iff

$$\int_{-\infty}^{0} a(t)\ dt \quad = \quad \infty \qquad (2)$$

As usual, "iff" denotes "if and only if". The proof is straightforwardly done by considering light–like geodesics in the $x-t$-plane. So, one directly concludes

that the statements do not depend on the number n of spatial dimensions, (only $n > 0$ is used), and the formulation for $n = 3$ was chosen as the most interesting case.

Moreover, lemma 2 remains valid if we replace "light–like geodesically complete" by "light–like and time–like geodesically complete".

In ref. 1 I wrote: "Lemma 2 seems to be unpublished up to now." Now I can add two earlier references [2], [3] which presented related statements: Borde, Vilenkin [2], besides considering more general inhomogeneous space-times, use metric (1) in conformal time η, i.e., $dt = a(\eta)d\eta$. Then we have instead of eq. (2) (see the appendix of [2]) the following condition

$$\int_{\eta_{\min}}^{\eta} a^2(\hat{\eta})\ d\hat{\eta} \quad = \quad \infty \tag{3}$$

Up to this time-reparametrization their result coincides with our lemma 2.

Romero, Sanchez [3], see also Sanchez [4], have discussed a quite general class of warped product space–times and the conditions of geodesic completeness in them, and our lemma 2 can be found after some tedious reformulations and specializations from Theorem 3.9. of [3].

Further recent results on singularity theorems are reviewed in Senovilla [5]. That review and most of the research concentrated mainly on the 4–dimensional Einstein theory. Probably, the majority of arguments can be taken over to the higher–dimensional Einstein equation without change, but this has not yet worked out up to now. And singularity theorems for $F(R)$-gravity are known up to now for very special cases only.

2 Rigorous Solutions

The most frequently used $F(R)$-Lagrangian is

$$L = \left(\frac{R}{2} \ - \ \frac{l^2}{12} R^2 \right) \ \sqrt{-g} \qquad \text{where } l > 0. \tag{4}$$

Here we discuss the non–tachyonic case only. From Lagrangian (4) one gets a fourth-order field equation. Only very few closed-form solutions ("exact solutions") are known. However, for the class of spatially flat Friedmann models, eq. (1), the set of solutions is qualitatively completely described (but not in closed form) in Müller, Schmidt [6]. We call them "rigorous solutions".

One of them, often called "Starobinsky inflation", can be approximated by eq. (1) with

$$a(t) \ = \ \exp(-\frac{t^2}{12l^2}) \tag{5}$$

This approximation is valid in the region $t \ll -l$. However, this solution does not fulfil the condition eq. (2). Therefore, by lemma 2, Starobinsky inflation does not represent a light–like geodesically complete cosmological model as has been frequently stated in the literature.

To prevent a further misinterpretation let me reformulate this result as follows: Inspite of the fact that the Starobinsky model is regular (in the sense that $a(t) > 0$ for arbitrary values of synchronized time t), every past–directed light–like geodesic terminates in a curvature singularity (i.e., $|R| \longrightarrow \infty$) at a finite value of its affine parameter. Therefore, the model is not only geodesically incomplete in the coordinates chosen, but it also fails to be a subspace of a complete space–time.

In contrast to this one can say: Eq. (1) with $a(t) = \exp(Ht)$, H being a positive constant, is the inflationary de Sitter space–time. According to lemma 2, it is also incomplete. However, contrary to the Starobinsky model, it is a subspace of a complete space–time, namely the de Sitter space-time represented as a closed Friedmann model.

For the Hamiltonian formulation of fourth-order gravity see e.g. Demaret, Querella [7] and Schmidt [8] and the cited references.

The case that the conformally invariant term $C_{ijkl}C^{ijkl}$ is also part of the gravitational Lagrangian has been quite often mentioned in the literature, e.g. as the most natural one besides Einstein gravity. On the other hand, besides ref. 6 only very few rigorous solutions are known for this case.

3 Dimensional Reduction

There exist several possibilites for a dimensional reduction of higher-dimensional space–times with some symmetries to a corresponding lower-dimensional space-time with additional fields; e.g. the 4-dimensional metrics with 2 commuting Killing vectors can be reduced to a 2-dimensional metric with 2 additional scalar fields. [9] Moreover, solutions of the corresponding gravity theories go over to solutions by this procedure.

For 2-dimensional gravity theories the closed–form solutions have been carefully studied recently, see e.g. Klösch, Strobl [10] and the references cited there.

So, even if there might be doubts about the physical significance of lower- or higher-dimensional gravity theories, one can apply the results of them (e.g. the presentation and classification of exact solutions) to give a mathematical relation to corresponding classes of 4–dimensional space-times.

Surprisingly, up to now this possibility to search for new solutions of Einstein's equation in 4 dimensions has not been applied very often.

4 Conformal Transformation

Besides dimensional reduction, the application of conformal transformations represents one of the most powerful methods to transform different theories into each other. In many cases this procedure related theories to each other which have been originally considered to be independent ones. The most often discussed question in this context is "which of the metrics is the physical one", but one can, of course, use such conformal relations also as a simple mathematical tool to find exact solutions without the necessity of answering this question. However, concerning a conformal transformation of singularity theorems one must be cautious, because typically, near a singularity in one of the theories, the conformal factor diverges, and then the conformally transformed metric need not be singular there.

For transformations of classes of theories containing $F(R)$-theories see e.g. [11] and Capozziello et al. [12], and for transformations of sixth–order theories stemming from $R\Box R$-Lagrangains see e.g. Wands [13].

5 Exact Solutions

A lot of papers claim to have presented new exact solutions of the Einstein equation (in arbitrary dimension), but in many of these cases the authors did not carefully check whether their solution is only a well-known old solution in unusual coordinates. To simplify this check, Mignemi, Schmidt [14] wrote down the n-dimensional de Sitter space–time in several classes of unusual coordinates. Maroto, Shapiro [15] and the book by Krasinski [16] represent further sources for methods to characterize exact cosmological solutions.

Finally, let me again stress the important difference of the Euclidean and the Lorentzian signature: Smooth connected complete Riemannian spaces are geodesically connected. However, smooth connected complete space–times (e.g. the de Sitter space–time) need not be geodesically connected (see for instance Sanchez [17]).

The topological origin of this distinction is the same as the distinction between lemma 1 and lemma 2: lemma 1 uses the compactness of the rotation group, whereas lemma 2 uses the noncompactness of the Lorentz group [18].

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